Unusual shapes for a catenary under the effects of surface tension and gravity: A variational treatment

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The familiar catenary is the shape assumed by a chain or string as it hangs from two points. The mathematical equation of the catenary was first published more than three hundred years ago by Leibnitz and Huygen, among others. Here we consider the shapes assumed by a hanging string in the presence of gravity and surface tension. The surface tension is introduced by suspending the string from a thin horizontal rod while the area bounded by the string and the rod is covered with a soap film. The string then assumes new and wonderful shapes depending on the relative strength of the surface tension and the weight per unit length of the string. When surface tension dominates, the string is pulled inward, assuming a convex shape similar to the Greek letter γ. On the other hand, when gravity is dominant the string is pulled outward and assumes a concave shape best described as a distorted catenary. However, when the gravitational force normal to the string matches the surface tension, the string takes a linear configuration similar to the letter *V*. Under suitable conditions, the string can be made to assume any of the three configurations by adjusting the separation of its end points. The equations that describe the shape of the string are derived by minimizing the total energy of the system and are presented for the three principal configurations.

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In late 17th century, the mathematical equation that describes the shape of a hanging chain or catenary (from the Latin word *catena* for chain), was published independently by several well known mathematicians of the time including Leibnitz, Huygen, and the two Bernoulli brothers Jakob and Johann [1].

Throughout the 18th century as the calculus of variations was being refined and extended, the hanging chain problem was often used to illustrate the power of a newly invented method. For example, in 1744 Euler cast the catenary in the form of an isoperimetric problem of finding a curve through two fixed points that gives the lowest center of gravity subject to a length constraint [1].

Here we consider the shapes assumed by a hanging string in the presence of gravity and surface tension. We introduce the surface tension by suspending the string from a thin horizontal rod while the area bounded by the string and the rod is covered with a soap film. The shape assumed by the string depends on the relative strength of the surface tension and the normal component of the gravitational force per unit length of the string. When surface tension dominates, the string is pulled inward, assuming a shape similar to the Green letter γ [Fig. 1(a)]. When gravity is dominant, the shape is that of a distorted catenary [Fig. 1(b)]. However, when the gravitational force normal to the string matches the surfaces tension, the string takes a linear configuration similar to the letter V [Fig. 1(c)]. Furthermore, under suitable conditions, the string can be made to assume any of the three configurations by adjusting the separation of its end points along the support rod.

We use the technique of the calculus of variations to derive the governing differential equations of the string by minimizing the total energy of the system (sum of the gravitational and surface energies) subject to a length constraint. The differential equations are then integrated to yield the mathematical equations of the string in closed form for the three principal configurations.

Figure 2 shows the right half of the string in a coordinate system where the string hangs from the x axis and the y axis forms a line of symmetry. The length of the string is $2L_0$, ds represents an element of the string, θ is the angle the string makes with the horizontal, and X_0 and Y_0 are defined in Fig. 2. In what follows, σ is the surface tension, λ is the linear mass density of the string, and g stands for the acceleration of gravity. The dimensionless ratio of $2\sigma/\lambda g$ will be denoted by α.

The total energy of the system can now be written as

$$U=\int_{-Y_0}^{0}[\lambda g y(1+x'^2)^{1/2}-2\sigma y x']dy \ , \tag{1}$$

where $x'=dx/dy$ and $(1+x'^2)^{1/2}dy=ds$. The first term in the integrand gives the gravitational potential energy relative to the x axis and the second term gives the surface energy. Note that in Eq. (1) since y is negative the surface energy term is positive and the gravitational potential energy relative to the x axis is negative as expected. Furthermore, the length constraint may be written as

$$L_0=\int_{-Y_0}^{0}(1+x'^2)^{1/2}dy \ . \tag{2}$$

To minimize the total energy U subject to this length constraint, we define $G(y,x,x')$ to be

$$G\equiv(\lambda g y+\gamma)(1+x'^2)^{1/2}-2\sigma y x' \ , \tag{3}$$

where γ is a Lagrange multiplier, and require that function G satisfy the Euler-Lagrange equation,

$$\frac{\partial G}{\partial x}=\frac{d}{dy}\left[\frac{\partial G}{\partial x'}\right]. \tag{4}$$

With our particular choice of variables, we note that G is not an explicit function of x. Therefore,

$$\frac{\partial G}{\partial x'}=C_0=x'(1+x'^2)^{-1/2}(\lambda g y+\gamma)-2\sigma y\ , \tag{5}$$

where C_0 is a constant.

Equation (5) is the differential equation of the system and may be solved to obtain $x(y)$. However, it is much more convenient to cast Eq. (5) into a parametric form by noting that $x'=\cot\theta$ and $(1+x')^{-1/2}=\sin\theta$. Hence, one can immediately write,

$$y=\frac{C_0-\gamma\cos\theta}{\lambda g\cos\theta-2\sigma}\ ,$$

which in turn can be simplified into,

$$\frac{y}{C}=\frac{1}{(\cos\theta-\alpha)}+C_y\ , \tag{6}$$

where as before $\alpha=2\sigma/\lambda g$, and C and C_y are given in terms of C_0 and γ by $C=(C_0-\gamma\alpha)/\lambda g$ and $C_y=\gamma/(\gamma\alpha-C_0)$.

Furthermore, Eq. (6) can be differentiated to give

$$\frac{dy}{d\theta}=\frac{C\sin\theta}{(\cos\theta-\alpha)^2}\ , \tag{7}$$

which immediately leads to

$$\frac{dx}{d\theta}=\frac{C\cos\theta}{(\cos\theta-\alpha)^2}\ , \tag{8}$$

because $dy/dx=\tan\theta$. Equations (7) and (8) constitute the parametric differential equations of the system. As noted before, Eq. (7) can be integrated easily to yield Eq. (6) which gives the functional dependence of y on θ. Similarly, Eq. (8) may be integrated to yield the functional dependence of x on θ. The final results depend on whether α is larger, equal to, or smaller than unity.

When $\alpha>1$, the surface tension dominates over gravity causing the string to assume a convex configuration [see Fig. 1(a)] with a clumped length l. The mathematical equations of the string are easily obtained by integrating Eqs. (7) and (8) to yield

$$\frac{x}{l}=\frac{-\alpha^2\sin\theta}{(\alpha^2-1)(\alpha-\cos\theta)} -\frac{2\alpha}{(\alpha^2-1)^{3/2}}\tan^{-1}\left[\frac{(\alpha+1)\tan(\theta/2)}{(\alpha^2-1)^{1/2}}\right]+C_x\ , \tag{9}$$

and

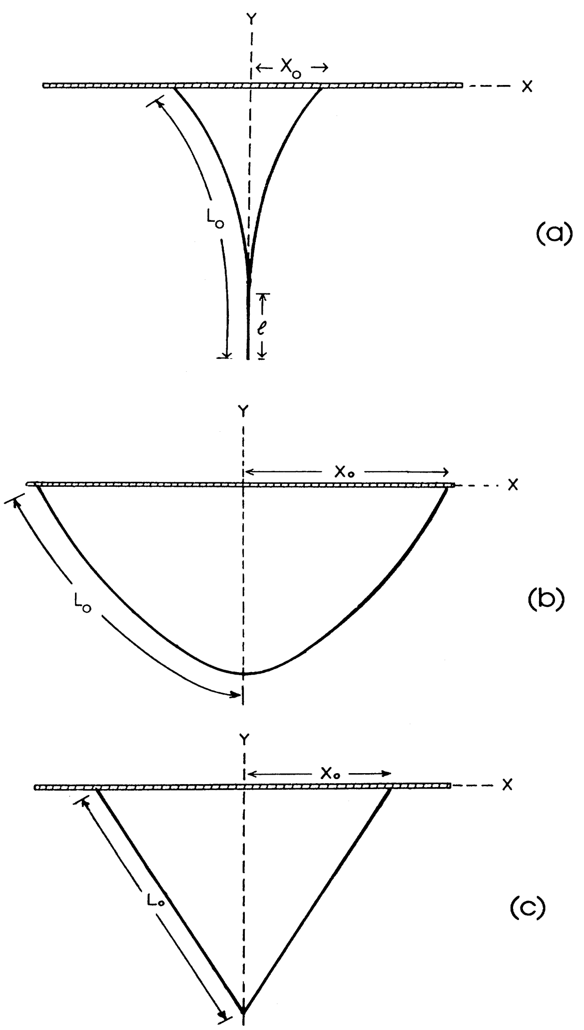

FIG. 1. The three principal configurations of the string. When surface tension is dominant, the string is pulled inward assuming the configuration shown in (a). When gravity is dominant, the string takes the form shown in (b). When the gravitational force component normal to the string, $\lambda g\cos\theta$, exactly balances the surface tension 2σ, then the string assumes the linear configuration shown in (c).

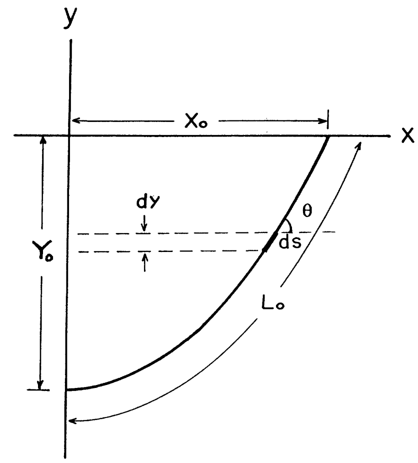

FIG. 2. The right half of the string as it hangs from the x axis. Note that the y axis forms a line of symmetry.

$$\frac{y}{l}=-\frac{\alpha}{\cos\theta-\alpha}+C_y \ , \tag{10}$$

where the constants l, C_x, and C_y can be found [2] in terms of α, X_0, and L_0.

When $\alpha=1$, the shape of the string remains convex [see Fig. 1(a)], but the equations of the string take a simpler form:

$$\frac{x}{l}=\frac{1}{6}\cot(\theta/2)[4-\sin^{-2}(\theta/2)]+C_x \ , \tag{11}$$

$$\frac{y}{l}=-[1/(\cos\theta-1)]+C_y \ . \tag{12}$$

When $\alpha<1$, three distinct configurations are possible [3] depending on the relative value of α and X_0/L_0. If $\alpha>X_0/L_0$, the configuration is convex similar to the case of $\alpha>1$ [Fig. 1(a)]. However, in this case, the equations of the string have a somewhat different functional form:

$$\frac{x}{l}=\frac{-\alpha^2\sin\theta}{(1-\alpha^2)(\cos\theta-\alpha)}+\frac{-\alpha}{(1-\alpha^2)^{3/2}}\ln\left|\frac{(1+\alpha)\tan(\theta/2)+\sqrt{1-\alpha^2}}{(1+\alpha)\tan(\theta/2)-\sqrt{1-\alpha^2}}\right|+C_x \ , \tag{13}$$

$$\frac{y}{l}=\frac{-\alpha}{\cos\theta-\alpha}+C_y \ . \tag{14}$$

When $\alpha=X_0/L_0$, the string assumes a linear configuration [Fig. 1(c)] given by

$$y=\frac{(L_0^2-X_0^2)^{1/2}}{X_0}(x-X_0) \ . \tag{15}$$

Finally, when $\alpha<X_0/L_0$, the configuration becomes concave best described as a distorted catenary [Fig. 1(b)], with the following equations:

$$\frac{x}{C}=\frac{\alpha\sin\theta}{(1-\alpha^2)(\cos\theta-\alpha)}+\frac{1}{(1-\alpha^2)^{3/2}}\ln\left|\frac{(1+\alpha)\tan(\theta/2)+\sqrt{1-\alpha^2}}{(1+\alpha)\tan(\theta/2)-\sqrt{1-\alpha^2}}\right| \ , \tag{16}$$

$$\frac{y}{C}=\frac{1}{\cos\theta-\alpha}-\frac{1}{\cos\theta_0-\alpha} \ , \tag{17}$$

where θ_0 is the angle the string makes with the x axis.

It is interesting to note that when $\alpha<1$, a string of length $2L_0$ can be made to assume a convex, linear, or concave configuration depending on the value of X_0 which is easily manipulated. Thus, starting with $\alpha=0.5$ and $L_0=2$, Fig. 3 shows the configurations of the right half of the string as X_0 is varied from 0.6 to 1.4. Note that the configuration is convex for $X_0<\alpha L_0$, becomes linear when $X_0=\alpha L_0$, and takes a concave configuration when $X_0>\alpha L_0$. The graphs of Fig. 3 were generated from numerical computations based on the equations of the string.

Experimentally, the change from a convex to concave

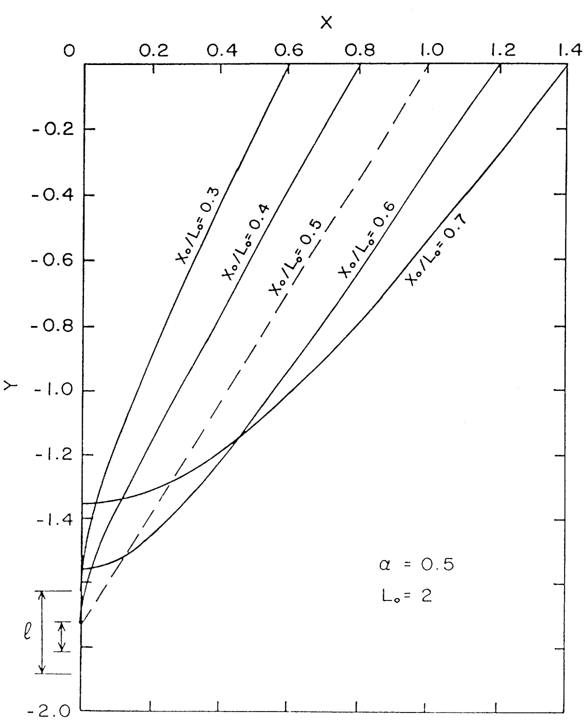

FIG. 3. The shapes assumed by the string (only the right half is shown) as X_0 is varied from 0.6 to 1.4 for a fixed value of $\alpha=0.5$, and $L_0=2$. These graphs are generated from numerical computations based on the equations developed in the text. Note that when $X_0=\alpha L_0=1$, the configuration is linear as shown in the dashed line. Above this line, the curves are convex with $X_0<\alpha L_0=1$, and below this line, the configurations are concave with $X_0>\alpha L_0=1$.

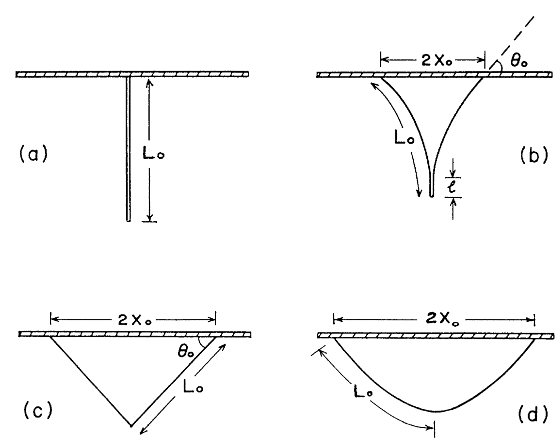

FIG. 4. The sequence of configurations assumed by the string as the ends' separation is increased. The configuration is a vertical line (a) when $X_0=0$, takes a convex form (b) when $X_0<\alpha L_0$, becomes linear (c) when $X_0=\alpha L_0$, and reverts to a distorted catenary (d) when $X_0>\alpha L_0$.

configuration is easily demonstrated by using a length of knitting yarn and a soap solution made of one part glycerin, two parts Joy (the dishwashing liquid), and seven parts distilled water. The two ends of the yarn are loosely tied to a glass rod to allow easy variation of the ends' separation. With the ends close together, the entire yarn is dipped into the soap solution [Fig. 4(a)]. Now when one end of the yarn is pulled away along the glass rod, the string begins with a convex configuration [Fig. 4(b)]. As X_0 increases the curvature moderates and the clumped length shortens until the linear configuration [Fig. 4(c)] is reached at $X_0=\alpha L_0$. The configuration becomes a distorted catenary [Fig. 4(d)] when X_0 is further increased. If X_0 is now reduced, the string retraces the entire sequence of configurations to return to its starting shape.

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